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A LINEAR APPROACH FOR SOLVING STOCHASTIC CAPITAL BUDGETING PROB--ETC(U)
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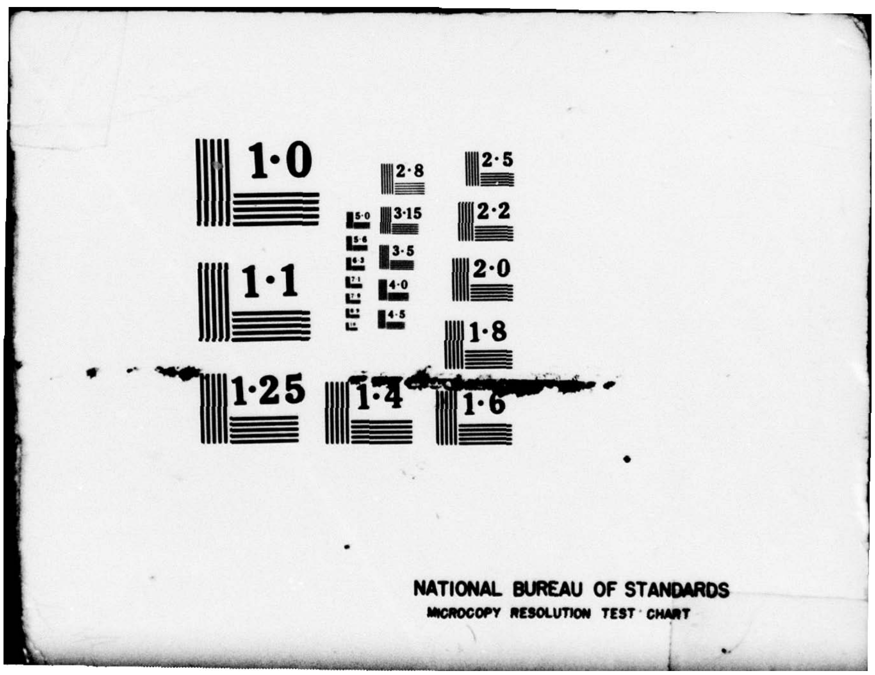
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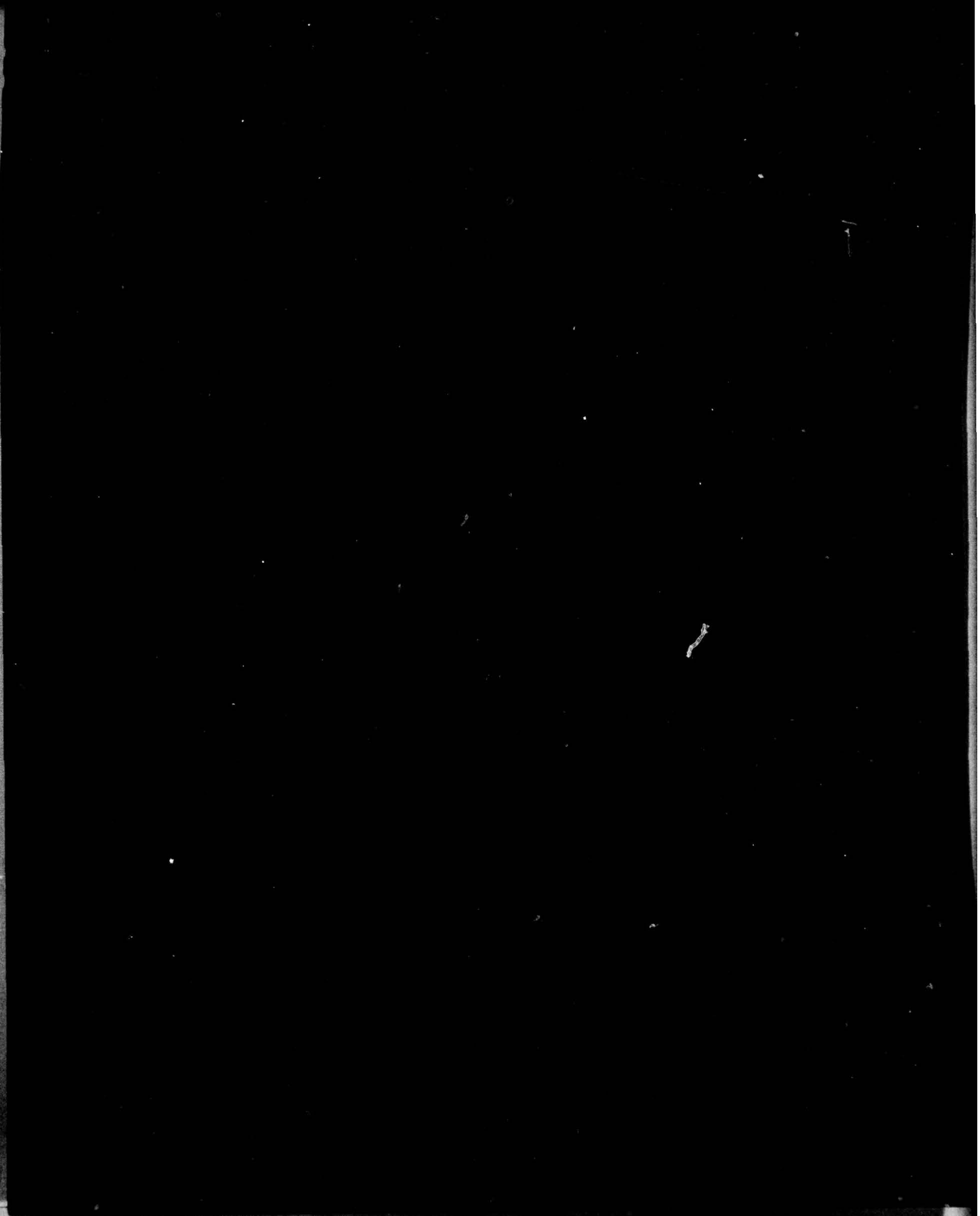

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A LINEAR APPROACH FOR SOLVING STOCHASTIC CAPITAL BUDGETING PROBLEMS.

by

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GERHARD SCHIEFER

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Frederick S. Hillier, Project Director

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1. Introduction

Considered is a capital budgeting problem with linear decision variables that can be either continuous or integer and where some or all of the associated cash flows are random variables that may be statistically dependent.

It is formulated as a problem of linear programming under risk for which mainly three approaches, stochastic programming, linear programming under uncertainty and chance-constrained programming, have been developed.[1)]

Stochastic linear programming, an approach originally suggested by Tintner [32] is primarily concerned with the probability distribution of the objective function's value.[2)]

Linear programming under uncertainty or, as it is sometimes termed, stochastic programming with recourse, was suggested independently by Beale [4] and Dantzig [10]. The basic innovation of this approach is to amend the problem to allow the decision maker the opportunity to make corrective actions after the random variables are observed and/or pay penalties for constraint violations.

Charnes, Cooper and Symonds [8] have presented a third approach called chance-constrained programming[3)] in which each constraint must be satisfied with a certain tolerance probability, i.e.,the fulfillment of the constraints is not required to be guaranteed as in linear programming under uncertainty.

For the formulation of objectives in chance-constrained programming approaches different criteria have been discussed, the principal ones outlined in Charnes and Cooper [7].

In this paper, the capital budgeting problem is considered to be adequately represented by a chance-constrained programming model with an objective function that includes the expected value and standard deviation of a firm's horizon value.

The application of chance-constrained programming to capital budgeting problems has been discussed before. See for example Arzac [1], Byrne, Charnes, Cooper and Kortanek [3], Byrne, Cooper, Charnes, Davis and Guilford [5], Hillier [15], Naslund [24,25], Naslund and Whinston [27], Robertson [28], Struve [31] or Spetzler [30].

The main purpose of this paper is to show that a stochastic capital budgeting problem with continuous and/or integer decision variables can be formulated as a convex chance-constrained programming problem that can be approximated by an ordinary (integer or noninteger) linear programming problem.

Furthermore, we will show that the proposed procedure allows the determination of an optimal cash reserve and, in addition, the consideration of decision opportunities that deal with an unexpected surplus or deficit in periodic capital budgets as well as of penalties involved if the probability constraints do happen to be violated.

2. A stochastic capital budgeting problem

2.1 Basic formulation

Let us consider a capital budgeting problem in which we assume that a firm chooses those physical and financial projects which maximize its horizon value subject to physical and financial constraints in distinguished periods.[4)] Furthermore, it is assumed that (a) interdependences between projects and (b) the capital market conditions can be expressed in an (integer or noninteger) linear programming format[5] such that the capital budgeting problem can be formulated mathematically as:

$$\text{maximize } z = \sum_{i=1}^{T} c_i x_i \qquad (1a)$$

$$\text{subject to } \sum_{i=1}^{t} g_{ti} x_i = -D_t \qquad (t=1,..,T) \qquad (1b)$$

$$x_i \in X \qquad (i=1,..,T) \qquad (1c)$$

$$0 \leq x_i \leq 1 \qquad (i=1,..,T) \qquad (1d)$$

where

x_i = vector of decision variables, each representing the fraction of a project (physical capital projects as well as borrowing and lending opportunities) started in period i,

c_i = vector representing the horizon value of post horizon cash flows associated with one unit of x_i,

g_{ti} = vector representing the net cash flow in period t associated with one unit of x_i,

D_t = the total amount of net cash flows in period t associated with projects realized prior to the start of the planning process,

and where the constraints $x_i \varepsilon X$ (i=1,..,T) can be expressed in an (integer or noninteger) linear programming format.[6]

The elements of the vectors c_i (i=1,..,T) and g_{ti} (t=1,..,T;i=1,..,t) and the D_t (t=1,..,T) are assumed to be either constants or random variables that might be statistically dependent. If some elements of the c_i or even some elements of the g_{ti} or some of the D_t are random variables, the horizon value z is also a random variable. The same is true for n_t (t=1,..,T), the net cash flow in period t, defined by

$$n_t = \sum_{i=1}^{t} g_{ti} x_i + D_t \qquad (t=1,..,T) \qquad (2)$$

if some elements of the g_{ti} or some of the D_t are random variables.

In this case, the problem formulation (1) must be regarded as an incomplete formulation of the decision situation as it does not reflect the decision maker's attitude towards the probability distributions of the horizon value and the periodic net cash flows. It may be completed by specifying rules that allow the determination of solutions for the decision problem which accord with the decision maker's preferences.

2.2 Decision rules

We consider here only rules for decision situations where we assume that the decision variables must be assigned values while the values the random variables will take on are still unknown (zero-order *decision rules*). For capital budgeting problems attempts to allow for the probability distribution of random variables have been concentrated on

a) imposing probability constraints on periodic net cash flows while maximizing the expected value of the random objective variable (see Hillier [15] or Naslund [25]) or

b) introducing an objective function which includes not only the expected value of the random objective variable but a measure for the dispersion of its probability distribution.[7)]

Both approaches have been extensively discussed by their authors with regard to the maximization of a decision maker's utility function. As for the choice of an appropriate dispersion measure in the second approach, comparative discussions and model calculations have been mainly concentrated on the variance, semi-variance and standard deviation of the objective variable's probability distribution (see Markowitz [21], Bey [2], Hamada [13], Mao [22,23], Robertson [28] or Van Horne [33,34]).

In this paper we combine both approaches as we assume a decision maker that allows for the probability distribution of the random periodic net cash flows n_t $(t=1,..,T)$ and the random horizon value z in the stochastic capital budgeting problem (1) by preferring decisions which are obtained according to the following rules:

1) In each period t $(t=1,..,T)$ the equality constraint (1b) should hold for the expected value $E(n_t)$ of the periodic net cash flow while the probability that a periodic deficit exceeds a specified amount b_t should be lower than a specified risk level $(1-\alpha_t)$, i.e.,

$$\begin{aligned} &E(n_t) = 0 \\ &\text{Prob}\,\{n_t \geq -b_t\} \geq \alpha_t \end{aligned} \tag{3}$$

2) A possible realization z' of the horizon value z, defined by

$$z' = E(z) - \beta\sqrt{V(z)} \quad \text{with} \quad \beta \geq 0 \tag{4}$$

with the expected value $E(z)$ and the standard deviation $\sqrt{V(z)}$ of the probability distribution of z, should be as high as possible. For a discussion and determination of β in a probability context see, for example, Hillier and Lieberman [16].

While the second rule focuses on the standard deviation of the horizon value's probability distribution as a dispersion measure to be used in the objective function of the programming model, it is easy to show that other measures like the variance or semi-variance of the distribution (as preferred by Markowitz [21]) can be used within the framework of the proposed procedure as well.

2.3 A complete formulation

By introducing the decision rules into the programming formulation (1) of the stochastic capital budgeting problem, it can be formulated as a chance-constrained programming problem:

$$\text{maximize} \quad z' = E\left(\sum_{i=1}^{T} c_i x_i\right) - \beta \sqrt{V\left(\sum_{i=1}^{T} c_i x_i\right)} \qquad (5a)$$

$$\text{subject to} \quad E\left(\sum_{i=1}^{t} g_{ti} x_i\right) = E(-D_t) \qquad (5b)$$

$$\text{Prob}\left\{\left(\sum_{i=1}^{t} g_{ti} x_i + D_t\right) \geq -b_t\right\} \geq \alpha_t \quad (t=1,..,T) \qquad (5c)$$

$$x_i \in X \quad (i=1,..,T) \qquad (5d)$$

$$0 \geq x_i \geq 1 \quad (i=1,..,T) \qquad (5e)$$

The first step in solving this problem is to reduce it to a deterministic equivalent form, i.e.,to determine deterministic equivalent forms of the probability constraints.

3. A deterministic equivalent form

Denote by s_t^k a "situation" k in period t, representing a possible simultaneous realization $[g_{1t}^k,..,g_{Tt}^k,D_t^k]$ of the random elements of g_{it} for i=1,..,T and D_t and by s_{T+1}^k a situation representing a simultaneous realization $[c_1^k,..,c_T^k]$ of the random elements of c_t (t=1,..,T). Let S_t be the set of all possible situations in a period t (t=1,..,T) or the horizon period T+1, respectively.

Furthermore, denote by s^j a sequence j of situations over the planning period such that

$$s^j = [s_1^j, s_2^j, .., s_{T+1}^j] \quad \text{with} \quad s_t^j \in S_t \quad (t=1,..,T+1)$$

and let S be the set of all possible sequences.

For the moment, it is assumed that S includes only a finite small number q of possible sequences and that these sequences and their probabilities p^j (j=1,..,q) are known. The possible net cash flows in the various periods and the possible horizon values are then determined by (a) the value of the decision vectors x_t (t=1,..,T) and (b) the possible realizations of situation sequences.

3.1 The probability constraints

Let us now consider the possible net cash flows in a single period t for a specified value of the decision vector x_i (i=1,..,t). Denote them by n^j_{tx} (j=1,..q) and regard them as the possible realizations of a random variable n_{tx}. Furthermore, denote by $E(n_{tx})$ the net cash flow in period t of an "expected situation sequence" representing the expected values $E(g_{ti})$, $E(D_t)$ and $E(c_i)$ of g_{ti}, D_t and c_i with i=1,..,T.

Define

$$k^{-j}_{tx} = \begin{Bmatrix} E(n_{tx}) - n^j_{tx} & \text{if } n^j_{tx} \leq E(n_{tx}) \\ 0 & \text{otherwise} \end{Bmatrix} \quad j=1,..,q \qquad (6a)$$

$$k^{+j}_{tx} = \begin{Bmatrix} n^j_{tx} - E(n_{tx}) & \text{if } n^j_{tx} > E(n_{tx}) \\ 0 & \text{otherwise} \end{Bmatrix} \quad j=1,..,q \qquad (6b)$$

Let us assume that the functional form of the probability distribution of n_{tx} is known and that the fractiles of this distribution are completely determined by its mean and variance.[8] Let F(u) denote the cumulative distribution function of the standardized variable $u = (n_{tx}-E(n_{tx}))/\sqrt{V(n_{tx})}$. Define $u_{\alpha t}$ by the relationship $F(u_{\alpha t})=\alpha_t$.

Then the deterministic equivalent form of the probability constraint in period t for a specified decision vector can be formulated as[9)]

$$E(n_{tx}) - u_{\alpha t}\sqrt{V(n_{tx})} \geq -b_t \tag{7}$$

However, if the functional form of the probability distribution of n_{tx} is not known, Tchebychev's inequality (or the extension of Camp-Meidel if n_{tx} can be regarded as unimodal, see Duncan [11]) still yields an upper bound for $u_{\alpha t}$.[10)] This bound is a very conservative measure. However, it will be shown that the model calculations yield some information about the functional form of the probability distribution of the net cash flow associated with the optimal decision vector. This information might be used (a) to adjust the assumption about the functional form or (b) to calculate the "true" α_t, i.e. the α_t which refers to the used $u_{\alpha t}$ and the updated information about the functional form of the net cash flow's probability distribution.

Whatever the probability distribution of n_{tx} happens to be, as its expected value $E(n_{tx})$ has been restricted to zero in (5), (7) can be rewritten as

$$u_{\alpha t}\sqrt{\sum_{j=1}^{q} p^j((k_{tx}^{-j})^2 + (k_{tx}^{+j})^2)} \leq b_t \qquad \text{or}$$

$$\sum_{j=1}^{q} p^j((k_{tx}^{-j})^2 + (k_{tx}^{+j})^2) \leq (b_t/u_{\alpha t})^2 \tag{8}$$

(8) defines a convex upper bound for the absolute deviation of the possible n_{tx}^j (j=1,..q) from $E(n_{tx})$. It restricts the set of feasible values of the decision vector x_i (i=1,..,t) in order to satisfy the probability constraints in (5).

As the value of the decision vector is not known explicitly, the values of the k_{tx}^{-j} (j=1,..,q) and k_{tx}^{+j} (j=1,..,q) in (8) are not either. However, they can be determined simultaneously with the determination of an optimal decision vector by introducing variables k_t^{-j} (j=1,,.,q) and k_t^{+j} (j=1,..,q) with the possible realizations

$$
\begin{aligned}
k_t^{-j} &: k_{tx}^{-j} \quad \forall\ x_i \in X \quad (i=1,,.,t) \\
k_t^{+j} &: k_{tx}^{+j} \quad \forall\ x_i \in X \quad (i=1,..,t)
\end{aligned}
\tag{9}
$$

into the programming model (5). The deterministic equivalent form of the probability constraint in period t for an unspecified decision vector can then be formulated as

$$\sum_{j=1}^{q} p^j((k_t^{-j})^2 + (k_t^{+j})^2) \leq (b_t/u_{\alpha t})^2 \tag{10}$$

From an economic point of view, these variables could be regarded as single period borrowing and lending opportunities for unexpected cash flows in period t. In this case, borrowing rates r_t^- and lending rates r_t^+ might apply such that the borrowing or lending of a cash unit in period t would affect the net cash flow in period t+1 by $R_t^-=(1+r_t^-)$ or $R_t^+=(1+r_t^+)$, respectively. While we will follow this interpretation when formulating the deterministic equivalent form of the chance-constrained programming problem, the variables could be considered alternatively as representing penalties involved if the constraints (5b) do happen to be violated. However, in this case the penalties might have to be paid in period t rather than in period t+1, i.e.,the model formulation would have to be changed accordingly.

3.2 Objective function

A similar procedure can be used to reduce the objective function of (5) to a more tractable form. Let us define a random variable w_x by the relationship

$$w_x = \sum_{i=1}^{T} c_i x_i^* - E\left(\sum_{i=1}^{T} c_i x_i^*\right) \tag{11}$$

where x_i^* (i=1,..,T) represents a specified value of the decision vector x_i (i=1,..,T). From (11) it follows that w_x has an expected value $E(w_x)=0$ and a variance

$$V(w_x) = V\left(\sum_{i=1}^{T} c_i x_i^*\right) \tag{12}$$

Now, let us denote the possible realizations of w_x by

$$w_x^{-j} = \begin{cases} E\left(\sum_{i=1}^{T} c_i x_i^*\right) - \sum_{i=1}^{T} c_i^j x_i^* & \text{if } c_i^j x_i^* \leq E\left(\sum_{i=1}^{T} c_i x_i^*\right) \\ 0 & \text{otherwise} \end{cases} \quad j=1,..,q$$

$$w_x^{+j} = \begin{cases} \sum_{i=1}^{T} c_i^j x_i^* - E\left(\sum_{i=1}^{T} c_i x_i^*\right) & \text{if } c_i^j x_i^* > E\left(\sum_{i=1}^{T} c_i x_i^*\right) \\ 0 & \text{otherwise} \end{cases} \quad j=1,..,q \tag{13}$$

where j marks the relevant situation sequence s^j (j=1,..,q). Then we get

$$V(w_x) = \sum_{j=1}^{q} p^j\left((w_x^{-j})^2 + (w_x^{+j})^2\right) \tag{14}$$

By proceeding as described above for the determination of the variance of a periodic net cash flow we introduce variables w^{-j} (j=1,..,q) and w^{+j} (j=1,..,q) with the possible realizations

$$
\begin{aligned}
w^{-j} &: \quad w_x^{-j} \quad \forall \; x_i \in X \qquad (i=1,..,T) \\
w^{+j} &: \quad w_x^{+j} \quad \forall \; x_i \in X \qquad (i=1,..,T)
\end{aligned}
\qquad (15)
$$

into the programming model. Let

$$y^2 = V(w^{-j}+w^{+j}) \quad \text{and} \quad v = E(\sum_{i=1}^{T} c_i x_i)$$

Then we can replace the objective function of the chance-constrained programming formulation by the equivalent

$$\text{maximize} \quad z' = v - \beta y$$

$$
\begin{aligned}
\text{subject to} \quad & E(\sum_{i=1}^{T} c_i x_i) - v = 0 \\
& \sum_{j=1}^{q} p^j((w^{-j})^2+(w^{+j})^2 - y^2 \leq 0
\end{aligned}
\qquad (16)
$$

where the quadratic constraint defines a convex bound for the set of feasible solutions of the decision problem.

3.3 Model

The incorporation of the variables k_t^{-j}, k_t^{+j}, w^{-j} and w^{+j} with $t=1,..,T$ and $j=1,..,q$ into the programming model requires the explicit consideration of each possible situation in period t ($t=1,..,T$) and the "horizon period" T+1, i.e. the consideration of all q situation sequences over the planning period. (This procedure is similar to the one used in stochastic programming with simple recourse [38]).

The deterministic equivalent form of the chance-constrained programming problem can then be formulated mathematically as:

$$\text{Maximize} \quad z' = v - \beta y$$

subject to

$$E(g_{11})x_1 = -E(D_1)$$

$$g_{11}^j x_1 + k_1^{-j} - k_1^{+j} = -D_1^j \qquad (j=1,..,q)$$

$$\sum_{j=1}^{q} (p^j(k_1^{-j})^2 + p^j(k_1^{+j})^2) \leq (b_1/u_{\alpha 1})^2$$

$$\sum_{i=1}^{t} E(g_{ti})x_i + \sum_{j=1}^{q} (-p^j R_{t-1}^- k_{t-1}^{-j} + p^j R_{t-1}^+ k_{t-1}^{+j}) = -E(D_t) \qquad (t=2,..,T)$$

$$\sum_{i=1}^{t} g_{ti}x_i - R_{t-1}^- k_{t-1}^{-j} + R_{t-1}^+ k_{t-1}^{+j} + k_t^{-j} - k_t^{+j} = -D_t^j \qquad (j=1,...,q;\ t=2,..,T)$$

$$\sum_{j=1}^{q} (p^j(k_t^{-j})^2 + p^j(k_t^{+j})^2) \leq (b_t/u_{\alpha t})^2 \qquad (t=2,..,T)$$

$$\sum_{i=1}^{T} E(c_i)x_i + \sum_{j=1}^{q} (-p^j R_T^- k_T^{-j} + p^j R_T^+ k_T^{+j}) - v = 0 \qquad (17)$$

$$\sum_{i=1}^{T} c_i^j x_i - R_T^- k_T^{-j} + R_T^+ k_T^{+j} - v + w^{-j} - w^{+j} = 0 \qquad (j=1,..,q)$$

$$\sum_{j=1}^{q} (p^j(w^{-j})^2 + p^j(w^{+j})^2) - y^2 \leq 0$$

$$x_i \in X \qquad (i=1,..,T)$$

$$0 \leq x_i \leq 1 \qquad (i=1,..,T)$$

$$k_t^{-j}, k_t^{+j}, v, w^{-j}, w^{+j}, y \geq 0 \qquad (j=1,..,q;\ t=1,..,T)$$

As the nonlinear constraints in (17) are all based on quadratic separable functions, they can easily be approximated by a grid linearization. In this case, efficient (integer or noninteger) linear programming algorithms can be used to solve the problem.

It is easy to show that the deterministic equivalent form of the chance-constrained programming problem can be adapted to more complex situations, i.e., situations where the decision maker's opportunity set for dealing with an unexpected periodic surplus or deficit consists of several distinguished single-period or multi-period borrowing and lending opportunities. Furthermore, a technically similar formulation would allow the consideration of penalties that have to be paid if a deficit in a period t exceeds specified amounts or if the probability constraints do happen to be violated.

As an example, consider a situation where several distinguished single-period borrowing and lending opportunities are available. Denote by γ_t^{-j} and γ_t^{+j} vectors that represent the borrowing and lending opportunities in a period t within the situation sequence j. Assume borrowing and lending rates to be represented by vectors R_t^- and R_t^+. Furthermore, assume that the capital market conditions for these opportunities can be expressed in a (integer or noninteger) linear programming format represented by the formulation $\gamma_t^{-j}, \gamma_t^{+j} \varepsilon \Lambda$.

Then the constraints in (17) that refer to a period t=2,..,T could be replaced by the following formulation:

$$
\begin{array}{ll}
\sum_{i=1}^{t} E(g_{ti})x_i + \sum_{j=1}^{q} (-p^j R^-_{t-1}\gamma^{-j}_{t-1} + p^j R^+_{t-1}\gamma^{+j}_{t-1}) & = -E(D_t) \\
\sum_{i=1}^{t} g_{ti}x_i \quad - R^-_{t-1}\gamma^{-j}_{t-1} + R^+_{t-1}\gamma^{+j}_{t-1} + k^{-j}_t \quad - k^{+j}_t & = -D^j_t \\
 & \quad (j=1,..,q) \\
\sum_{j=1}^{q} (p^j (k^{-j}_t)^2 + p^j (k^{+j}_t)^2) & \leq (b_t/u_{\alpha t})^2 \qquad (18) \\
+\gamma^{-j}_t \quad - \gamma^{+j}_t - k^{-j}_t \quad + k^{+j}_t & = 0 \\
 & \quad (j=1,..,q) \\
\gamma^{-j}_t, \gamma^{+j}_t & \varepsilon\ \Lambda \\
 & \quad (j=1,..,q)
\end{array}
$$

4.2 Consideration of a variable cash reserve

The b_t (t=1,..,T) in the constraints (5c) can be regarded as restrictions on the amount of cash that is easily available from external sources for the compensation of "unexpected" deficits in the periodic net cash flows, i.e., their negative deviation from the expected values. The formulation (5) does not allow the consideration of internal resources for this purpose. This is due to the assumption usually made in chance-constrained programming approaches for capital budgeting problems (see, for example, Naslund [25]) that the decision variables represent "contracts" that cannot be changed during a period for compensating unexpected deficits.

However, the possibility of determining an optimal periodic cash reserve from internal resources can be incorporated into the programming formulations (5) and (17).

Let us assume that external and internal resources are available for compensating unexpected deficits in a period t. Denote by $\bar{y}_t$ a decision variable that represents the amount reserved from internal resources. Then (5b) and (5c) in (5) can be replaced by

$$\begin{aligned} &E\left(\sum_{i=1}^{t} g_{ti}x_i\right) - \bar{y}_t = E(-D_t) && (t=1,..,T) \\ &\text{Prob}\left\{\left(\sum_{i=1}^{t} g_{ti}x_i + D_t\right) \geq -(b_t+\bar{y}_t)\right\} \geq \alpha_t && (t=1,..,T) \qquad (19) \\ &\bar{y}_t \geq 0 && (t=1,..,T) \end{aligned}$$

For reducing the probability constraints in (19) to a deterministic equivalent form we define variables $\overset{*}{y}_t$ by the relationship

$$\overset{*}{y}_t = b_t + \bar{y}_t \qquad (t=1,..,T) \qquad (20)$$

and proceed as described in the preceding sections. We then get

$$\begin{aligned} &u_{\alpha t} \sum_{j=1}^{q} \left(p^j (k_t^{-j})^2 + (k_t^{+j})^2\right) - (\overset{*}{y}_t)^2 \leq 0 && (t=1,..,T) \\ &\bar{y}_t - \overset{*}{y}_t \geq -b_t && (t=1,..,T) \end{aligned} \qquad (21)$$

The quadratic constraints that define convex bounds on the set of feasible solutions of the problem are based on quadratic separable functions, i.e.,their introduction into the programming model (17) would not affect the possibility of its approximation by a linear programming formulation.

5. Random sampling of situation sequences

Suppose now that the number q of possible situation sequences is large and that it is not feasible or economical to consider all sequences explicitly in the model formulation. In this case, the decision of the decision maker might be based on a random sample of sequences.

Assuming that the future probability distributions of the random variables and their interdependences are known, a random sample of situation sequences could be generated by a simulation process. However, the available data about future events is often restricted to a number of sequences usually derived from the past, i.e., from time series, etc. Then this set of sequences is usually regarded to represent a random sample.

However a random sample of situation sequences happens to have been formulated, its use affects the formulation of the probability constraints of the problem. If the probability distributions of the periodic net cash flows (or the horizon value) can be regarded as normal or approximately normal, the necessary changes in the formulation of the deterministic equivalent forms of the probability constraints can be calculated in a straightforward manner.

Let us consider a probability constraint in the form (7) and neglect for simplification the indices x and t such that

$$E(n) - u_\alpha \sqrt{V(n)} \geq -b \qquad (22)$$

From the sample mean $\bar{m}$ and the sample variance s^2 we may compute $(\bar{m}-u_\alpha s)$ as an estimate of $(E(n)-u_\alpha\sqrt{V(n)})$. In repeated sampling from a stable distribution this estimate will be approximately normally distributed (see Hald [12]) about $(E(n)-u_\alpha\sqrt{V(n)})$ so that we cannot be sure that $(\bar{m}-u_\alpha s)$ as computed from a single sample will be larger or equal $(-b)$ even if $E(n)-u_\alpha\sqrt{V(n)} \geq -b$. To be "reasonably" sure, i.e.,with a specified probability p, that the probability constraints will hold, the multiplication factor u_α in $(\bar{m}-u_\alpha s)$ has to be replaced by a factor u_β such that

$$\text{Prob}\,\{(\bar{m}-u_\beta s) \geq (E(n)-u_\alpha\sqrt{V(n)})\} \geq p \tag{23}$$

It has been shown elsewhere (see Hald [12]) that the normal distribution that approximates the probability distribution of $\lambda=(\bar{m}-u_\beta s)$ has mean $M(\lambda)=(E(n)-u_\beta\sqrt{V(n)})$ and variance

$$V(\lambda) = V(n)\left(\frac{1}{d} - \frac{u_\beta^2}{2f}\right) \tag{24}$$

where d denotes the sample size and $f=(d-1)$ the degree of freedom. Now, let $F(u)$ denote the cumulative distribution function of the standardized variable $u=(\lambda-M(\lambda))/\sqrt{V(\lambda)})$. Define u_p by the relationship $F(u_p)=p$. Then by proceeding in the usual way, the probability expression (23) can be replaced by

$$M(\lambda)-u_p\sqrt{V(\lambda)} \geq (E(n)-u_\alpha\sqrt{V(n)} \tag{25}$$

or

$$-u_\beta - u_p\sqrt{\frac{1}{d} - \frac{u_\beta^2}{2f}} \geq -u_\alpha \tag{26}$$

As u_p, u_α, d and f are assumed to be known, u_β can be calculated from (26).

If the functional form of the probability distributions of the net cash flows n_t $(t=1,..,T)$ are not normal, the sampling distributions of means and variances will not be the same as if they were normal, but the computations will not be very seriously affected unless the departure from normality is very marked and the samples are small (see Duncan [11]).

If the functional form is not known or we are unwilling to make assumptions regarding it, then it is still possible to derive the mean and variance of $\lambda=(\bar{m}-u_\beta s)$ from the sample (see Duncan [11]). But no information is available about the functional form of the probability distribution of λ because no assumptions can be made about the functional form of the distribution of s. In this case one might apply Tchebychev's inequality to get some rough limits on the probability variation.

6. Conclusion

In the preceding sections we were considering a capital budgeting problem with continuous and/or integer decision variables where some or all elements of the data were statistically dependent or independent random variables. It was formulated as a chance-constrained programming problem that allowed the explicit consideration of

a) decision opportunities dealing with a deficit or surplus in periodic net cash flows such as the accumulation of an optimal cash reserve or appropriate borrowing and lending opportunities and of

b) penalties that have to be paid if periodic deficits do occur or the probability constraints do happen to be violated.

It could be shown that the chance-constrained programming problem can be approximated by an ordinary (integer or noninteger) linear programming problem. The proposed procedure is based on the explicit consideration of possible realizations of the random variables in the programming formulation as discussed in linear programming under uncertainty for stochastic programs with simple recourse. The resulting increase in the size of the programming formulation could be limited by basing the procedure only on those possible realizations of the random variables that were connected with a random sample of situation sequences, i.e., of possible subsequent periodic realizations of the random variables over the planning period.

Footnotes

1. For a survey see Naslund [25], Vajda [36] or Kall [18].
2. See Sengupta [29] for a comprehensive discussion.
3. See also Charnes and Cooper [6].
4. See Charnes, Cooper and Miller [9] for the original presentation of a horizon value objective function and Weingartner [37] for a comparative discussion of models using the horizon value or the present value of a firm in their objective function, respectively.
5. See Weingartner [37] for some specific formulations.
6. Note that the constraints (1b) are formulated as equalities. This is due to the assumption that the set of bending and borrowing opportunities in each period t includes all available alternatives for cash use.
7. See Markowitz [21] for the original presentation though in the context of a portfolio investment problem.
8. See Hillier [15] for a discussion of relations between the probability distribution of the random variables g_{ti} $(i=1,..,t)$ and D_t and the probability distribution of their linear combination n_{tx} and the conditions under which the probability distribution of n_{tx} can be regarded as normal or at least (by some version of the Central Limit Theorem) approximately normal.
9. See Hillier and Lieberman [16].
10. See Hillier [14].

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A LINEAR APPROACH FOR SOLVING
STOCHASTIC CAPITAL BUDGETING PROBLEMS

Gerhard Schiefer

We ~~consider~~ capital budgeting problems with linear decision variables that can be either continuous or integer and where some or all of the associated cash flows are random variables that may be statistically dependent are considered. They are formulated as convex chance-constrained programming problems that can be approximated by ordinary (integer or non-integer) linear programming problems. The proposed procedure allows the explicit consideration of decision opportunities dealing with a deficit or surplus in periodic net cash flows such as the accumulation of an optimal cash reserve or appropriate borrowing and lending opportunities and of penalties that have to be paid if periodic deficits do occur or the probability constraints do happen to be violated.